Approximate Current Changes Needed In Booster Quadrupoles For A Given Tune Change

C.Y. Tan

Fermi National Accelerator Laboratory,

P.O. Box 500, Batavia, IL 60510-5011, USA.

(Dated: May 16, 2014)

The goal is to find an approximate formula that relates the current change needed in the `QL` and `QS` quadrupoles to the tune change requested by the user. The secondary goal is to verify that these equations are the same as those found in `B15` and `B84`.

I. INTRODUCTION

I will use the FODO lattice formulæ for calculating the current change that needs to be sent to the `QL` and `QS` corrector quadrupoles given the user requested tune change. Note that the Booster lattice is really FOFDOOD that comes from the combined function magnets and have nothing to do with the `QL`'s or `QS`'s. I think that this approximation is probably not that good but this has been used forever and so I will just use it.

II. THEORY

The approximation that I will use is two fold. They are:

- Replace the FOFDOOD cell with a FODO cell.
- Add `QS` to just before the F quadrupole and `QL` to just before the D quadrupole in the FODO cell. In fact, when I examine the `MADX` description of Booster, `QS` is before the short FD and `QL` is before the long DF section of each period the lattice. Note that both `QS` and `QL` are wired to be horizontal focusing quadrupoles.

The approximation that I have discussed above is illustrated below in Eq. 1

$$FOFDOOD \quad \rightarrow \quad \mathtt{QS}(\mathrm{FO})\mathtt{QL}(\mathrm{DO}) \tag{1}$$

And for Booster, there are 24 such $\texttt{QS}$(FO)$\texttt{QL}$(DO) periods.

Therefore, in this model, I can use the results that is derived in Appendix A to show that when I make a small tune change in either plane (dQ_H, dQ_V), the required $\texttt{QS}$ and $\texttt{QL}$ current needed is given by

$$\begin{pmatrix} dQ_H \\ dQ_V \end{pmatrix} = \frac{N}{4\pi} \begin{pmatrix} \beta_{\texttt{QS}H} & \beta_{\texttt{QL}H} \\ -\beta_{\texttt{QS}V} & -\beta_{\texttt{QL}V} \end{pmatrix} \begin{pmatrix} \frac{1}{B\rho}\left[\frac{\Delta B'L}{\Delta I}\right] I_{\texttt{QS}} \\ \frac{1}{B\rho}\left[\frac{\Delta B'L}{\Delta I}\right] I_{\texttt{QL}} \end{pmatrix} \tag{2}$$

where $N = 24$ is the number of $\texttt{QS}$(FO)$\texttt{QL}$(DO) periods, $\beta_{\texttt{QS}H}$ and $\beta_{\texttt{QS}V}$ are the values of the horizontal and vertical β's at the $\texttt{QS}$ quadrupole respectively. Similarly for the β's at the $\texttt{QL}$ quadrupole. Note that $\texttt{QS}$ is at the same location as the F quadrupole and so its β values are the same as that of the F quadrupole. Similarly for the $\texttt{QL}$ and the D quadrupole. Continuing, $B\rho$ is the magnetic rigidity of the beam, $[\Delta B'L/\Delta I]$ is the integrated focusing strength of the quadrupole per ampere in units of $(\mathrm{T}\cdot\mathrm{m})/(\mathrm{m}\cdot\mathrm{A}) = \mathrm{T/A}$, $I_{\texttt{QS}}$ is the current that needs to be supplied to the $\texttt{QS}$ quadrupoles and $I_{\texttt{QL}}$ for the $\texttt{QL}$ quadrupoles.

Note: In principle, $[\Delta B'L/\Delta I]$ can have different values for $\texttt{QL}$ and $\texttt{QS}$, but at least in the Booster, they have the same value and is 2.489×10^{-3} T/A.

It is obvious that there is a common factor in Eq. 2, and so I can define a new variable Γ that takes the following form

$$\Gamma \equiv \frac{N}{4\pi}\frac{1}{B\rho}\left[\frac{\Delta B'L}{\Delta I}\right] \tag{3}$$

When I substitute this into Eq. 2, I get

$$\begin{pmatrix} dQ_H \\ dQ_V \end{pmatrix} = \Gamma \begin{pmatrix} \beta_{\texttt{QS}H} & \beta_{\texttt{QL}H} \\ -\beta_{\texttt{QS}V} & -\beta_{\texttt{QL}V} \end{pmatrix} \begin{pmatrix} I_{\texttt{QS}} \\ I_{\texttt{QL}} \end{pmatrix} \tag{4}$$

Thus, I can invert the matrix in Eq. 4 to obtain $(I_{\texttt{QS}}, I_{\texttt{QL}})$ in terms of (dQ_H, dQ_V)

$$\begin{pmatrix} I_{\texttt{QS}} \\ I_{\texttt{QL}} \end{pmatrix} = \frac{1}{\Gamma(\beta_{\texttt{QS}V}\beta_{\texttt{QL}H} - \beta_{\texttt{QS}H}\beta_{\texttt{QL}V})} \begin{pmatrix} -\beta_{\texttt{QL}V} & -\beta_{\texttt{QL}H} \\ \beta_{\texttt{QS}V} & \beta_{\texttt{QS}H} \end{pmatrix} \begin{pmatrix} dQ_H \\ dQ_V \end{pmatrix} \tag{5}$$

A. Example

Now, suppose I want to change the vertical tune by dQ_V but I want to keep $dQ_H = 0$. Substituting this into Eq. 5, I get

$$I_{\mathtt{QS}} = -\frac{\beta_{\mathtt{QL}H} \times dQ_V}{\Gamma(\beta_{\mathtt{QS}V}\beta_{\mathtt{QL}H} - \beta_{\mathtt{QS}H}\beta_{\mathtt{QL}V})}$$
$$I_{\mathtt{QL}} = +\frac{\beta_{\mathtt{QS}H} \times dQ_V}{\Gamma(\beta_{\mathtt{QS}V}\beta_{\mathtt{QL}H} - \beta_{\mathtt{QS}H}\beta_{\mathtt{QL}V})} \tag{6}$$

It's interesting that the horizontal β's determine the currents required in the corrector quadrupoles for a vertical tune change!

III. MAPPING β'S TO LONG AND SHORT SECTIONS AND ALEX'S PROGRAM

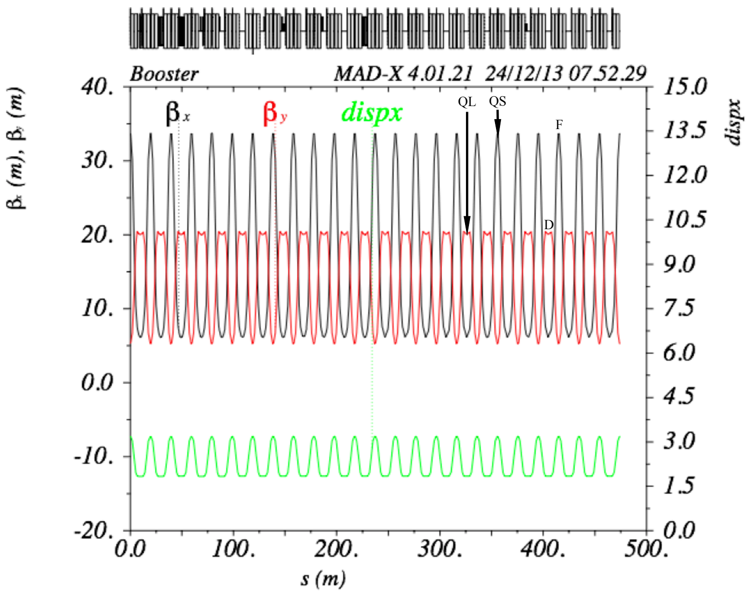

FIG. 1. The ideal Booster lattice with all the `QL`'s and `QS`'s set to zero. Because of space constraints, I can only indicate the location of the one of the F and D quadrupoles in this picture. Suffice to say that there is a F at every β_x peak and D at every β_y peak. And in my model, `QS` is at the same location as the F quadrupole and `QL` is at the same location as the D quadrupoles. But recall that both `QS` and `QL` are horizontal focusing quadrupoles.

When I look at the ideal lattice shown in Fig. 1, I can map the β's in the matrix of

Eq. 2 by placing F quadrupoles at the locations where β_x is at its maximum value and D quadrupoles at the locations where β_y is at its maximum value. Therefore, the mapping is as follows:

$$\begin{aligned}
\beta_{\mathtt{QL}V} &= 20.4245 \text{ m} \\
\beta_{\mathtt{QL}H} &= 7.2985 \text{ m} \\
\beta_{\mathtt{QS}V} &= 5.2871 \text{ m} \\
\beta_{\mathtt{QS}H} &= 33.6431 \text{ m}
\end{aligned} \tag{7}$$

A. Mapping β's to Alex's B15 program

I will map what Alex uses in his program `B15` to what I have derived. Looking at his source code in `CorrectorQuads.cpp`, I have the following

Variable name		Alex's variable name
$\Gamma \times B\rho$	$\rightarrow$	`commonFactor`
$\beta_{\mathtt{QL}V}$	$\rightarrow$	`betaVLong`
$\beta_{\mathtt{QL}H}$	$\rightarrow$	`betaHLong`
$\beta_{\mathtt{QS}V}$	$\rightarrow$	`betaVShort`
$\beta_{\mathtt{QS}H}$	$\rightarrow$	`betaHShort`

The matrix in Eq. 4 when mapped to what Alex uses in his program is

$$\begin{aligned}
\Gamma \begin{pmatrix} \beta_{\mathtt{QS}H} & \beta_{\mathtt{QL}H} \\ -\beta_{\mathtt{QS}V} & -\beta_{\mathtt{QL}V} \end{pmatrix} &\rightarrow \frac{\mathtt{commonFactor}}{B\rho} \begin{pmatrix} \mathtt{betaHShort} & \mathtt{betaHLong} \\ -\mathtt{betaVShort} & -\mathtt{betaVLong} \end{pmatrix} \\
&= \frac{1}{B\rho} \begin{pmatrix} \mathtt{b} & \mathtt{a} \\ \mathtt{d} & \mathtt{c} \end{pmatrix} \equiv \boldsymbol{M}
\end{aligned} \tag{8}$$

Inverting Alex's matrix $\boldsymbol{M}$ gives me

$$\boldsymbol{M}^{-1} = \frac{B\rho}{\mathtt{ad} - \mathtt{bc}} \begin{pmatrix} -\mathtt{c} & \mathtt{a} \\ \mathtt{d} & -\mathtt{b} \end{pmatrix} \tag{9}$$

and Alex's solution is

$$\begin{pmatrix} I_{\mathtt{QS}} \\ I_{\mathtt{QL}} \end{pmatrix} = \frac{B\rho}{\mathtt{ad} - \mathtt{bc}} \begin{pmatrix} -\mathtt{c} & \mathtt{a} \\ \mathtt{d} & -\mathtt{b} \end{pmatrix} \begin{pmatrix} dQ_H \\ dQ_V \end{pmatrix} \tag{10}$$

1. Verifying the equations in B15 and B84

Like in the example that I had discussed in section II A, I want to change the vertical tune dQ_V but not dQ_H and so

$$I_{\mathtt{QS}} = \frac{\mathtt{a}}{\mathtt{ad} - \mathtt{bc}} \times dQ_V \times B\rho \tag{11}$$

$$I_{\mathtt{QL}} = \frac{-\mathtt{b}}{\mathtt{ad} - \mathtt{bc}} \times dQ_V \times B\rho \tag{12}$$

And similarly when I want to change the horizontal tune dQ_H but not dQ_V, I get

$$\begin{aligned} I_{\mathtt{QS}} &= \frac{-\mathtt{c}}{\mathtt{ad} - \mathtt{bc}} \times dQ_H \times B\rho \\ I_{\mathtt{QL}} &= \frac{\mathtt{d}}{\mathtt{ad} - \mathtt{bc}} \times dQ_H \times B\rho \end{aligned} \tag{13}$$

These are exactly the equations used in `B15` and `B84` for calculating the required current changes for a given betatron tune change. Therefore, I have verified that the equations are correct in these programs.

IV. ACKNOWLEDGEMENTS

I would like to thank M. McAteer for pointing out that all the `QS` and `QL` corrector quadrupoles are wired as horizontal focusing quadrupoles.

Appendix A: Deriving the tune change from a perturbative lens

The Twiss matrix of a FODO cell (for example, see Ref. [1]) is given by

$$T_{\mathrm{FODO}} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \tag{A1}$$

where α, β, γ and μ have their usual definitions. When I add in a thin lens at the start of the FODO cell, I have

$$T'_{\mathrm{FODO}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \tag{A2}$$

where $1/f = \left[\frac{\Delta B' L}{\Delta I}\right] I/B\rho$. When I multiply out Eq. A2, I get

$$T'_{\mathrm{FODO}} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\frac{1}{f}(\cos\mu + \alpha\sin\mu) - \gamma\sin\mu & -\frac{\beta}{f}\sin\mu + \cos\mu - \alpha\sin\mu \end{pmatrix} \tag{A3}$$

With the inclusion of the perturbation, the phase advance μ changes and I write the change as $\mu \to \mu + d\mu = \mu'$. But for small enough $d\mu$, the α, β and γ should still be the same and so Eq. A1 becomes

$$T'_{\mathrm{FODO}} = \begin{pmatrix} \cos\mu' + \alpha\sin\mu' & \beta\sin\mu' \\ -\gamma\sin\mu' & \cos\mu' - \alpha\sin\mu' \end{pmatrix} \tag{A4}$$

When I equate Eq. A3 to A4, I have

$$\begin{aligned} &\begin{pmatrix} \cos\mu' + \alpha\sin\mu' & \beta\sin\mu' \\ -\gamma\sin\mu' & \cos\mu' - \alpha\sin\mu' \end{pmatrix} \\ &= \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\frac{1}{f}(\cos\mu + \alpha\sin\mu) - \gamma\sin\mu & -\frac{\beta}{f}\sin\mu + \cos\mu - \alpha\sin\mu \end{pmatrix} \end{aligned} \tag{A5}$$

I can judiciously select out the entries in Eq. A5 to work with. In this case, the $(1,1)$ and $(2,2)$ elements of the matrices can be equated to give

$$\begin{aligned} \cos\mu' + \alpha\sin\mu' &= \cos\mu + \alpha\sin\mu \\ \cos\mu' - \alpha\sin\mu' &= -\frac{\beta}{f}\sin\mu + \cos\mu - \alpha\sin\mu \end{aligned} \tag{A6}$$

And when I add these two equations together, I get

$$2\cos\mu' = 2\cos\mu - \frac{\beta}{f}\sin\mu \tag{A7}$$

I can expand Eq. A7 to first order in $d\mu$ to get

$$
\begin{aligned}
2(\cos\mu - d\mu\sin\mu) &= 2\cos\mu - \frac{\beta}{f}\sin\mu \\
\Rightarrow \qquad d\mu &= \frac{\beta}{2f}
\end{aligned}
\tag{A8}
$$

But $dQ = d\mu/2\pi$ and thus when I substitute this into Eq. A8, I obtain

$$
dQ = \frac{\beta}{4\pi f}
\tag{A9}
$$

a. *Orthogonal plane effects*

Now that I have Eq. A9, I can calculate what happens to the betatron tune when the corrector quadrupole current is changed. Both planes are affected because the quadrupole focuses in one plane and defocuses in the other.

For an F quadrupole (both `QS` and `QL` quadrupoles are F quadrupoles) that has focal length f_F and in this case, the change in the betatron tunes of both planes is

$$
\begin{aligned}
dQ_H &= \frac{\beta_{FH}}{4\pi f_F} \\
dQ_V &= -\frac{\beta_{FV}}{4\pi f_F}
\end{aligned}
\tag{A10}
$$

where β_{FH} and β_{FV} are the values of the horizontal and vertical β's at the F quadrupole respectively.

Therefore, when I apply Eq. A10 to one `QS`(FO)`QL`(DO) cell to take into account the effect of both `QS` and `QL`, I get by summing both their effects to obtain

$$
\begin{aligned}
dQ_H &= \frac{\beta_{\mathtt{QS}H}}{4\pi f_{\mathtt{QS}}} + \frac{\beta_{\mathtt{QL}H}}{4\pi f_{\mathtt{QL}}} \\
dQ_V &= -\frac{\beta_{\mathtt{QS}V}}{4\pi f_{\mathtt{QS}}} - \frac{\beta_{\mathtt{QL}V}}{4\pi f_{\mathtt{QL}}}
\end{aligned}
\tag{A11}
$$

where $f_{\mathtt{QS}}$ is the focal length of the `QS` quadrupole and $\beta_{\mathtt{QS}H}$ and $\beta_{\mathtt{QS}V}$ are the values of the horizontal and vertical β's at the `QS` quadrupole respectively. The variables for the `QL` quadrupole are named similarly.

I can write Eq. A11 in matrix form

$$
\begin{aligned}
\begin{pmatrix} dQ_H \\ dQ_V \end{pmatrix} &= \frac{1}{4\pi} \begin{pmatrix} \beta_{\mathtt{QS}H} & \beta_{\mathtt{QL}H} \\ -\beta_{\mathtt{QS}V} & -\beta_{\mathtt{QL}V} \end{pmatrix} \begin{pmatrix} \frac{1}{f_{\mathtt{QS}}} \\ \frac{1}{f_{\mathtt{QL}}} \end{pmatrix} \\
&= \frac{1}{4\pi} \begin{pmatrix} \beta_{\mathtt{QS}H} & \beta_{\mathtt{QL}H} \\ -\beta_{\mathtt{QS}V} & -\beta_{\mathtt{QL}V} \end{pmatrix} \begin{pmatrix} \frac{1}{B\rho}\left[\frac{\Delta B'L}{\Delta I}\right] I_{\mathtt{QS}} \\ \frac{1}{B\rho}\left[\frac{\Delta B'L}{\Delta I}\right] I_{\mathtt{QL}} \end{pmatrix}
\end{aligned} \tag{A12}
$$

Notice that I have explicitly put in the `QS` current $I_{\mathtt{QS}}$ and `QL` current $I_{\mathtt{QL}}$.

When the same currents $I_{\mathtt{QS}}$ and $I_{\mathtt{QL}}$ are applied to every `QS`(FO)`QL`(DO) cell in the lattice and there are N cells, then I just sum all the tune change from every cell to arrive at

$$
\begin{pmatrix} dQ_H \\ dQ_V \end{pmatrix} = \frac{N}{4\pi} \begin{pmatrix} \beta_{\mathtt{QS}H} & \beta_{\mathtt{QL}H} \\ -\beta_{\mathtt{QS}V} & -\beta_{\mathtt{QL}V} \end{pmatrix} \begin{pmatrix} \frac{1}{B\rho}\left[\frac{\Delta B'L}{\Delta I}\right] I_{\mathtt{QS}} \\ \frac{1}{B\rho}\left[\frac{\Delta B'L}{\Delta I}\right] I_{\mathtt{QL}} \end{pmatrix} \tag{A13}
$$

Thus, Eq. A13 is the formula that I will use in section II.

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[1] H.Wiedemann. page 245. Springer, 3rd edition, 2007.